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Reconstruction of the standard model in a generalized differential geometry based on the real structure

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Abstract. The standard model is reconstructed in a generalized differential geometry (GDG) based on the idea of a real structure as proposed by Coquereaux et al. and Connes. The GDG considered in this article is a kind of non-commutative geometry (NCG) on the discrete space that successfully reproduces the Higgs mechanism of the spontaneously broken gauge theory. Here, a GDG is a direct generalization of the differential geometry on an ordinary continuous manifold to the product space of this manifold with a discrete manifold. In a GDG, a one-form basis χ on the discrete space is incorporated in addition to the one-form basis $\mathrm{d}x^\mu$ on Minkowski space, rather than γ^5 as in Connes's original work. Although the Lagrangians obtained in this way are the same as those obtained in our previous formulation of GDG, the basic formalism becomes very simply and clear.

1 Introduction

The reconstruction of spontaneously broken gauge theory in non-commutative geometry (NCG) on the discrete space $M_4 \times Z_2$ (in general, $M_4 \times Z_N$) has elucidated the essence of the Higgs mechanism. It revealed that the Higgs boson field is a kind of gauge field to be considered as a connection on the discrete space. This approach was initiated by Connes [1] in 1990, and since then many authors have studied NCG on the discrete space and proposed[2–22] many formulation variants.

We also proposed the χ formalism [15] in which we introduce the one-form basis χ on the discrete space Z_2 in addition to the ordinary one-form base $\mathrm{d}x^\mu$ on the Minkowski space M_4 to describe the generalized gauge field containing the gauge and Higgs boson fields. This formulation of NCG is very similar to the differential geometry on a continuous manifold. Thus, we call our formulation a generalized differential geometry (GDG). Based on this formulation, and adopting the fermion field of the $SO(10)$ grand unified theory (GUT), the standard model was reconstructed [21] to yield, making reasonable assumptions, the mass relation between the weak gauge boson and the Higgs boson, together with the Weinberg angle in the $SO(10)$ GUT. However, in this construction, the method of introducing the color gauge field is rather awkward. The purpose of this paper is to modify this method based on the idea of a real structure proposed by Coquereaux et al. and Connes [3] who introduced it to incorporate the color gauge symmetry and explain the anomaly-free condition.

Though the same subject was treated in our recent paper [23], the NCG formulation given there is rather different from that given in the present paper, especially with respect to the representation of the fermion field and the algebraic rule of the one-form χ. For this reason, the present study is useful for the purpose of obtaining a better understanding of the most natural mathematical structure underlying the standard model.

This article is divided into four sections. Section 2 presents the modifications of our previous formalism based on the real structure operator obtained by introducing the anti-fermion field in addition to the fermion field of the $SO(10)$ GUT. In this section, a geometrical picture for the unification of the gauge and Higgs fields is realized, which is the ultimate understanding in the NCG approach. Section 3 represents an application to the reconstruction of the standard model. Although the Lagrangians obtained in this way are the same as those in our previous formulation of NCG, the basic formulation becomes very simple and clear. Section 4 is devoted to concluding remarks.

2 Basic formulation

The reformulation of the GDG presented in our previous scheme [21] is performed by taking account of the idea of the real structure proposed by Coquereaux et al. and Connes [3]. The real structure operator J defined in this

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paper is hereafter explained in accordance with the GDG formulation.

In [21], the fermion fields $\psi(x,y)$ with arguments x and y ($y=+,-$) on the product space $M_4\times Z_2$ are taken as

$$\psi(x,+)=\frac{1}{\sqrt{2}}\begin{pmatrix} u^{\rm r}_{\rm L}\\ u^{\rm g}_{\rm L}\\ u^{\rm b}_{\rm L}\\ \nu_{\rm L}\\ d^{\rm r}_{\rm L}\\ d^{\rm g}_{\rm L}\\ d^{\rm b}_{\rm L}\\ e_{\rm L}\end{pmatrix},\quad \psi(x,-)=\frac{1}{\sqrt{2}}\begin{pmatrix} u^{\rm r}_{\rm R}\\ u^{\rm g}_{\rm R}\\ u^{\rm b}_{\rm R}\\ \nu_{\rm R}\\ d^{\rm r}_{\rm R}\\ d^{\rm g}_{\rm R}\\ d^{\rm b}_{\rm R}\\ e_{\rm R}\end{pmatrix},\quad (2.1)$$

where the subscripts L and R stand for the left-handed and right-handed fermions, respectively, and the superscripts r, g and b represent the color indices. We must consider three generations in the standard model. Therefore, in precise notation, for example, u should be written as $(u,c,t)^{\rm t}$ with the superscript t defining the transpose of the matrix. In this notation, $\psi(x,y)$ is a vector in 24-dimensional Hilbert space. In order to incorporate the real structure into the GDG formulation, the fermion space $\mathcal{H}$ is doubled into the direct sum of two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^{\rm c}$ representing particles and anti-particles, respectively. The anti-particles $\psi^{\rm c}(x,y)$ in $\mathcal{H}^{\rm c}$ are represented as

$$\psi^{\rm c}(x,+)=\frac{1}{\sqrt{2}}\begin{pmatrix} d^{\rm r\,c}_{\rm L}\\ d^{\rm g\,c}_{\rm L}\\ d^{\rm b\,c}_{\rm L}\\ e_{\rm L}{}^{\rm c}\\ -u^{\rm r\,c}_{\rm L}\\ -u^{\rm g\,c}_{\rm L}\\ -u^{\rm b\,c}_{\rm L}\\ -\nu_{\rm L}{}^{\rm c}\end{pmatrix},\quad \psi^{\rm c}(x,-)=\frac{1}{\sqrt{2}}\begin{pmatrix} d^{\rm r\,c}_{\rm R}\\ d^{\rm g\,c}_{\rm R}\\ d^{\rm b\,c}_{\rm R}\\ e_{\rm R}{}^{\rm c}\\ -u^{\rm r\,c}_{\rm R}\\ -u^{\rm g\,c}_{\rm R}\\ -u^{\rm b\,c}_{\rm R}\\ -\nu_{\rm R}{}^{\rm c}\end{pmatrix}. \quad (2.2)$$

The role of J is to interchange ψ and $\psi^{\rm c}$, and thus it is essentially a charge conjugation operator. In precise notation, J in the present formulation is

$$J=C\sigma^2\otimes\sigma^2\otimes 1^4\otimes 1^3, \quad (2.3)$$

where σ^2 is a Pauli matrix and C is the charge conjugation operator. If $\Psi(x,y)$ is written as

$$\Psi(x,y)=\begin{pmatrix}\psi(x,y)\\ \psi^{\rm c}(x,y)\end{pmatrix}, \quad (2.4)$$

the operation of J in (2.3) on $\Psi(x,y)$ yields

$$J\Psi(x,y)=\Psi(x,y). \quad (2.5)$$

Thus, the fermion field $\Psi(x,y)$ is invariant under the real structure operator J.

The structure of the Yang–Mills–Higgs (YMH) sector is determined in correspondence with that of the fermion space $\mathcal{H}\oplus\mathcal{H}^{\rm c}$. Following the reconstruction of the standard model by Coquereaux et al. and Connes, the fundamental functions $\tilde{a}_i(x,y)$ and $\tilde{b}_i(x,y)$ are written as

$$\tilde{a}_i(x,y)=\begin{pmatrix} a_i(x,y) & 0\\ 0 & c^{\rm c}_i(x,y)\end{pmatrix},\quad \tilde{b}_i(x,y)=\begin{pmatrix} b_i(x,y) & 0\\ 0 & c^{\rm c}_i(x,y)\end{pmatrix}, \quad (2.6)$$

where $c^{\rm c}_i(x,y)$ is defined as

$$c^{\rm c}_i(x,y)=J'c_i(x,y)(J')^\dagger=\sigma^2\otimes 1^4\otimes 1^3 c^*_i(x,y)\sigma^2\otimes 1^4\otimes 1^3, \quad (2.7)$$

with the operator $J'={\rm i}C\sigma^2\otimes 1^4\otimes 1^3$ and the asterisk $*$ indicating complex conjugation. The matrix-valued functions $a_i(x,y)$ and $c_i(x,y)$ are the constituent functions of the flavor and color gauge fields, respectively, and these functions are expressed in 24×24 matrix form, corresponding to the representations in (2.1) and (2.2).

The operation of the real structure operator on the functions $\tilde{a}(x,y)$ and $\tilde{b}(x,y)$ is given, using the operator J in (2.3), by

$$J\tilde{a}_i(x,y)J^\dagger=\begin{pmatrix} c_i(x,y) & 0\\ 0 & a^{\rm c}_i(x,y)\end{pmatrix},\quad J\tilde{b}_i(x,y)J^\dagger=\begin{pmatrix} c_i(x,y) & 0\\ 0 & b^{\rm c}_i(x,y)\end{pmatrix}. \quad (2.8)$$

With the functions $\tilde{a}_i(x,y)$ and $\tilde{b}_i(x,y)$, we can simplify the definition of the generalized gauge field $\mathcal{A}(x,y)$ to

$$\mathcal{A}(x,y)=\frac{1}{2}\sum_i\left(\tilde{b}^\dagger_i(x,y)\mathbf{d}\tilde{a}_i(x,y)+\tilde{a}^\dagger_i(x,y)\mathbf{d}\tilde{b}_i(x,y)\right), \quad (2.9)$$

where the subscript i is required to insure that the gauge fields included in (2.9) are realistic, as shown later. $\mathbf{d}$ in (2.9) is the generalized exterior derivative defined as follows:

$$\begin{aligned}&\mathbf{d}={\rm d}+{\rm d}_\chi,\\ &{\rm d}f(x,y)=\partial_\mu f(x,y){\rm d}x^\mu, \quad (2.10)\\ &{\rm d}_\chi f(x,y)=[-f(x,y)\tilde{M}(y)+\tilde{M}(y)f(x,-y)]\chi,\end{aligned}$$

where $f(x,y)$ represents the function $\tilde{a}_i(x,y)$ or $\tilde{b}_i(x,y)$, ${\rm d}x^\mu$ is the ordinary one-form basis, taken to be dimensionless, in Minkowski space M_4, and χ is the one-form basis, also assumed to be dimensionless, in the discrete space Z_2. Here we have introduced the x-independent matrix $\tilde{M}(y)$ which is written according to the representation in (2.6) as

$$\tilde{M}_i(y)=\begin{pmatrix} M(y) & 0\\ 0 & 0\end{pmatrix}. \quad (2.11)$$

The matrix $M(y)$ appearing here turns out to determine the scale and pattern of the spontaneous breakdown of the gauge symmetry in the flavor sector, and its hermitian conjugation is given by $M(y)^\dagger = M(-y)$. Equation (2.11) indicates that the color symmetry of the strong interaction does not break spontaneously.

In order to find explicit forms of the gauge and Higgs fields according to (2.9) and (2.10), we need the following important algebraic rule of non-commutative geometry:

$$f(x,y)\chi = \chi f(x,-y), \tag{2.12}$$

where $f(x,y)$ represents any field defined on the discrete space such as $\tilde{a}_i(x,y)$, the gauge field, the Higgs field or the fermion fields. It should be noted that (2.12) never expresses the relation between the matrix elements of $f(x,+)$ and $f(x,-)$ but insures a consistent product between the fields expressed in differential form on the discrete space. Equation (2.12) characterizes a non-commutative geometry in the present formulation. Using (2.10) and (2.12), $\mathcal{A}(x,y)$ can be rewritten as

$$\mathcal{A}(x,y) = \tilde{A}_\mu(x,y)\mathrm{d}x^\mu + \tilde{\Phi}(x,y)\chi, \tag{2.13}$$

where

$$\tilde{A}_\mu(x,y) = \begin{pmatrix} A_\mu(x,y) & 0 \\ 0 & G^{\mathrm{c}}_\mu(x,y) \end{pmatrix}, \tag{2.14}$$

$$\tilde{\Phi}(x,y) = \begin{pmatrix} \Phi(x,y) & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.15}$$

$A_\mu(x,y)$, $\Phi(x,y)$ and $G_\mu(x)$ are expressed using the constituent fields $a_i(x,y)$ and $c_i(x,y)$ through

$$A_\mu(x,y) = \frac{1}{2}\sum_i \Big(b_i^\dagger(x,y)\partial_\mu a_i(x,y) + a_i^\dagger(x,y)\partial_\mu b_i(x,y)\Big), \tag{2.16}$$

$$\Phi(x,y) = \frac{1}{2}\sum_i \Big\{b_i^\dagger(x,y)\left(-a_i(x,y)M(y) + M(y)a_i(x,-y)\right) + a_i^\dagger(x,y)\left(-b_i(x,y)M(y) + M(y)b_i(x,-y)\right)\Big\}, \tag{2.17}$$

$$G_\mu(x,y) = \sum_i c_i^\dagger(x,y)\partial_\mu c_i(x,y), \tag{2.18}$$

and they are identified with the gauge field in the flavor sector, the Higgs field, and the color gauge field of the strong interaction, respectively.

In order to identify $A_\mu(x,y)$ and $G_\mu(x)$ with true gauge fields, the following conditions have to be imposed:

$$\sum_i b_i^\dagger(x,y)a_i(x,y) = 1^{24}, \tag{2.19}$$

$$\sum_i c_i^\dagger(x)c_i(x) = \frac{1}{g_s}1^{24}. \tag{2.20}$$

Here, g_s is a constant related to the corresponding coupling constant as specified below. In general, we can set the right-hand side of (2.19) equal to $1/g_f$. However, we do not do this simply to avoid the complexity.

According to (2.8), the generalized gauge field $\tilde{A}(x,y)$ is transformed under the real structure operation as

$$J\tilde{A}(x,y)J^\dagger = J\begin{pmatrix} A(x,y) & 0 \\ 0 & G^{\mathrm{c}}(x,y) \end{pmatrix}J^\dagger = \begin{pmatrix} G(x,y) & 0 \\ 0 & A^{\mathrm{c}}(x,y) \end{pmatrix}, \tag{2.21}$$

where

$$A(x,y) = A_\mu(x,y)\mathrm{d}x^\mu + \Phi(x,y)\chi, \tag{2.22}$$

$$G(x,y) = G_\mu(x,y)\mathrm{d}x^\mu. \tag{2.23}$$

Equation (2.21) enables us to obtain the gauge invariant Dirac Lagrangian, as shown below. It should be noted that $(\mathrm{d}x^\mu)^* = \mathrm{d}x^\mu$ and $\chi^* = \chi$.

Before constructing the gauge covariant field strength, we address the gauge transformations of $\tilde{a}_i(x,y)$ and $\tilde{b}_i(x,y)$, which are defined as

$$\tilde{a}_i^g(x,y) = \tilde{a}_i(x,y)\tilde{g}(x,y), \quad \tilde{b}_i^g(x,y) = \tilde{b}_i(x,y)\tilde{g}(x,y). \tag{2.24}$$

Here, the gauge transformation function $\tilde{g}(x,y)$ is expressed corresponding to (2.6) as

$$\tilde{g}(x,y) = \begin{pmatrix} g_f(x,y) & 0 \\ 0 & g_c^{\mathrm{c}}(x,y) \end{pmatrix}, \tag{2.25}$$

where $g_f(x,y)$ and $g_c(x,y)$ are the gauge functions with respect to the corresponding flavor unitary group and the color $\mathrm{SU}(3)_c$ group, respectively. It should be noticed that $g_f(x,y)$ and $g_c(x)$ can both be chosen so as to commute with each other, so that $g_c(x)$ commutes with $a_i(x,y)$, $b_i(x,y)$ and $M(y)$, and $g_f(x,y)$ commutes with $c_i(x)$. Then we obtain the gauge transformation of $\mathcal{A}(x,y)$:

$$\tilde{\mathcal{A}}^{\mathrm{g}}(x,y) = \tilde{g}^{-1}(x,y)\tilde{\mathcal{A}}(x,y)\tilde{g}(x,y) + \tilde{g}^{-1}(x,y)\begin{pmatrix} 1^{24} & 0 \\ 0 & \frac{1}{g_s}1^{24} \end{pmatrix}\mathbf{d}\tilde{g}(x,y), \tag{2.26}$$

where use has been made of (2.9) and (2.20), and as in (2.10),

$$\mathbf{d}\tilde{g}(x,y) = \partial_\mu\tilde{g}(x,y)\mathrm{d}x^\mu + [-\tilde{g}(x,y)\tilde{M}(y) + \tilde{M}(y)\tilde{g}(x,-y)]\chi. \tag{2.27}$$

Using (2.26)–(2.31), we find the gauge transformations of the gauge and Higgs fields as

$$A^{\mathrm{g}}_\mu(x,y) = g_f^{-1}(x,y)A_\mu(x,y)g_f(x,y) + g_f^{-1}(x,y)\partial_\mu g_f(x,y), \tag{2.28}$$

$$\Phi^{\mathrm{g}}(x,y) = g_f^{-1}(x,y)\Phi(x,y)g_f(x,-y) + g_f^{-1}(x,y)\partial_y g_f(x,y), \tag{2.29}$$

$$G^{\mathrm{g}}_\mu(x) = g_c^{-1}(x)G_\mu(x)g_c(x) + \frac{1}{g_s}g_c^{-1}(x)\partial_\mu g_c(x), \tag{2.30}$$

where we define the operator ∂_y as

$$\partial_y g_f(x,y) = [-g_f(x,y)M(y) + M(y)g_f(x,-y)]. \quad (2.31)$$

Equation (2.29) is very similar to the other two equations, (2.28) and (2.30), and so it strongly indicates that the Higgs field is a kind of gauge field on the discrete space $M_4 \times Z_2$. From (2.31), (2.29) can be rewritten as

$$\Phi^{\mathrm{g}}(x,y) + M(y) = g_f^{-1}(x,y)(\Phi(x,y) + M(y))g_f(x,-y), \quad (2.32)$$

which makes it obvious that

$$H(x,y) = \Phi(x,y) + M(y) \quad (2.33)$$

is an un-shifted Higgs field whereas $\Phi(x,y)$ denotes a shifted one with vanishing vacuum expectation value.

In addition to the algebraic rules in (2.10) we add an important rule:

$$\mathrm{d}_\chi \tilde{M}(y) = \tilde{M}(y)\tilde{M}(-y)\chi \quad (2.34)$$

which yields, together with (2.10), the nilpotency of the generalized exterior derivative $\mathbf{d}$. The nilpotency of $\mathbf{d}$ is explained in detail in [21]. With these considerations, we define the gauge covariant field strength as

$$\begin{aligned}\mathcal{F}(x,y) &= \mathbf{d}\mathcal{A}(x,y) \\ &\quad + \mathcal{A}(x,y)\begin{pmatrix} 1^{24} & 0 \\ 0 & g_s 1^{24}\end{pmatrix} \wedge \mathcal{A}(x,y) \\ &= \begin{pmatrix} F(x,y) & 0 \\ 0 & \mathcal{G}^{\mathrm{c}}x,y)\end{pmatrix}, \end{aligned} \quad (2.35)$$

where the field strengths of the flavor and color gauge fields are defined by

$$\begin{aligned} F(x,y) &= \mathbf{d}A(x,y) + A(x,y)\wedge A(x,y), \\ \mathcal{G}(x,y) &= \mathrm{d}G(x,y) + g_s G(x,y)\wedge G(x,y). \end{aligned} \quad (2.36)$$

Owing to the nilpotency of $\mathbf{d}$, we can easily derive the gauge transformation of $\mathcal{F}(x,y)$ to find it to be

$$\mathcal{F}^g(x,y) = \tilde{g}^{-1}(x,y)\mathcal{F}(x,y)\tilde{g}(x,y). \quad (2.37)$$

The algebraic rules defined in (2.10) and (2.34) and the definition (2.36) yield

$$\begin{aligned} F(x,y) &= \frac{1}{2}F_{\mu\nu}(x,y)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\ &\quad + D_\mu\Phi(x,y)\mathrm{d}x^\mu \wedge \chi + V(x,y)\chi\wedge\chi, \end{aligned} \quad (2.38)$$

where

$$\begin{aligned} F_{\mu\nu}(x,y) &= \partial_\mu A_\nu(x,y) - \partial_\nu A_\mu(x,y) \\ &\quad +[A_\mu(x,y), A_\mu(x,y)], \\ D_\mu\Phi(x,y) &= \partial_\mu\Phi(x,y) + A_\mu(x,y)H(x,y) \\ &\quad -H(x,y)A_\mu(x,-y) \\ V(x,y) &= (\Phi(x,y)+M(y))(\Phi(x,-y)+M(-y)) \\ &\quad -Y(x,y). \end{aligned} \quad (2.39)$$

$Y(x,y)$ in this equation is an auxiliary field expressed as

$$\begin{aligned} Y(x,y) &= \frac{1}{2}\sum_i \Big(b_i^\dagger(x,y)M(y)M(-y)a_i(x,y) \\ &\quad + a_i^\dagger(x,y)M(y)M(-y)b_i(x,y)\Big), \end{aligned} \quad (2.40)$$

which becomes a constant field in the present construction of the standard model. In contrast to the complicated form of $F(x,y)$, $\mathcal{G}(x)$ is simply written

$$\begin{aligned} \mathcal{G}(x) &= \frac{1}{2}G_{\mu\nu}(x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\ &= \frac{1}{2}\{\partial_\mu G_\nu(x) - \partial_\nu G_\mu(x) + g_s[G_\mu(x), G_\mu(x)]\} \\ &\quad \times \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \end{aligned} \quad (2.41)$$

With the same metric structure on the discrete space $M_4 \times Z_2$ as that used in [21] we obtain the gauge invariant Yang–Mills–Higgs Lagrangian:

$$\begin{aligned} \mathcal{L}_{\mathrm{YMH}}(x) &= -\mathrm{Tr}\sum_{y=\pm}\frac{1}{g_y^2}\langle \mathcal{F}(x,y), \mathcal{F}(x,y)\rangle \\ &= -\mathrm{Tr}\sum_{y=\pm}\frac{1}{2g_y^2}F^\dagger_{\mu\nu}(x,y)F^{\mu\nu}(x,y) \\ &\quad + \mathrm{Tr}\sum_{y=\pm}\frac{1}{g_y^2}(D_\mu\Phi(x,y))^\dagger D^\mu\Phi(x,y) \\ &\quad - \mathrm{Tr}\sum_{y=\pm}\frac{1}{g_y^2}V^\dagger(x,y)V(x,y) \\ &\quad - \mathrm{Tr}\sum_{y=\pm}\frac{1}{2g_y^2}G^\dagger_{\mu\nu}(x)G^{\mu\nu}(x), \end{aligned} \quad (2.42)$$

where g_y is a constant related to the coupling constant of the flavor gauge field, and Tr denotes the trace over internal symmetry matrices including the color and flavor symmetries and the generation space. The third term on the right-hand side is the potential term of the Higgs particle.

Let us turn to the fermion sector to construct the Dirac Lagrangian. We begin by defining the covariant derivative acting on the spinor field $\Psi(x,y)$ in (2.4) which is the representation of the semi-simple group of the corresponding flavor and color gauge group. This covariant derivative is defined by using the real structure operator J:

$$\mathcal{D}\Psi(x,y) = (\mathbf{d} + \tilde{A}^D(x,y) + J\tilde{A}^D(x,y)J^\dagger)\Psi(x,y), \quad (2.43)$$

which we call the covariant spinor one-form, and $\tilde{A}^D(x,y)$ is chosen to make $\mathcal{D}\Psi(x,y)$ gauge covariant. $\tilde{A}^D(x,y)$ consists of two parts, the gauge and Higgs boson fields, namely

$$\tilde{A}^D(x,y) = \tilde{A}^D_\mu(x,y)\mathrm{d}x^\mu + \tilde{\Phi}^D(x,y)\chi, \quad (2.44)$$

where the gauge boson part $\tilde{A}^D_\mu(x,y)$ is defined as

$$\tilde{A}^D_\mu(x,y) = \begin{pmatrix} 1^{24} & 0 \\ 0 & g_s 1^{24}\end{pmatrix}\tilde{A}_\mu(x,y) \quad (2.45)$$

and the Higgs boson part $\tilde{\Phi}^D(x,y)$ is defined as

$$\tilde{\Phi}^D(x,+) = \tilde{\Phi}(x,+)\tilde{g}_{\rm Y},$$
$$\tilde{\Phi}^D(x,-) = (\tilde{\Phi}^D(x,+))^\dagger = \tilde{g}_{\rm Y}^\dagger \tilde{\Phi}(x,-). \qquad (2.46)$$

Here, $\tilde{g}_{\rm Y} = {\rm diag}(g_{\rm Y},0)$ is the Yukawa coupling constant later described in precise form. Since the role of d_χ in (2.38) is to induce the shift $\Phi(x,y) \to \Phi(x,y) + M(y)$ as shown previously, we define its action on a fermion field analogously as

$$d_\chi \Psi(x,y) = \tilde{K}(y)\chi\Psi(x,y) = \tilde{K}(y)\Psi(x,-y)\chi, \qquad (2.47)$$

where $\tilde{K}(+) = \tilde{M}(+)\tilde{g}_Y$ and $\tilde{K}(-) = \tilde{K}(+)^\dagger$ corresponding to the representation in (2.46). With these considerations, (2.43) is rewritten as

$$\mathcal{D}\Psi(x,y) = (\partial_\mu + \tilde{A}_\mu(x,y) + J\tilde{A}_\mu(x,y)J^\dagger)\Psi(x,y){\rm d}x^\mu$$
$$+ (\tilde{H}^D(x,y) + J\tilde{H}^D(x,y)J^\dagger)\Psi(x,-y)\chi, \qquad (2.48)$$

where

$$\tilde{H}^D(x,+) = \left(\tilde{\Phi}(x,+) + \tilde{M}(+)\right)\tilde{g}_{\rm Y},$$
$$\tilde{H}^D(x,-) = \tilde{H}^D(x,+)^\dagger, \qquad (2.49)$$

from which

$$H^D(x,+) = (\Phi(x,+) + M(+))\, g_{\rm Y},$$
$$H^D(x,-) = H^D(x,+)^\dagger \qquad (2.50)$$

follow. Here, we assume that the Yukawa coupling constant g_Y commutes with the gauge function $g_f(x,-)$ so that the gauge covariance of the field $H^D(x,y)$ is kept:

$$H^D(x,y)^{\rm g} = g_f(x,y)H^D(x,y)g_f(x,-y). \qquad (2.51)$$

This assumption is satisfied in the reconstruction of the standard model in the next section. It should be noted that, owing to the real structure operator J, we have two equations,

$$\tilde{A}_\mu(x,y) + J\tilde{A}_\mu(x,y)J^\dagger = \begin{pmatrix} A_\mu(x,y) + G_\mu(x,y) & 0 \\ 0 & A^{\rm c}_\mu(x,y) + G^{\rm c}_\mu(x,y) \end{pmatrix}, \qquad (2.52)$$

$$\tilde{H}^D(x,y) + J\tilde{H}^D(x,y)J^\dagger = \begin{pmatrix} H^D(x,y) & 0 \\ 0 & H^{D^{\rm c}}(x,y) \end{pmatrix}, \qquad (2.53)$$

which insure the gauge covariance of the covariant spinor one-form $\mathcal{D}\Psi(x,y)$ under the gauge transformation of $H^D(x,y)$ given in (2.51). That is, as $\Psi(x,y)$ is subject to the gauge transformation

$$\Psi^{\rm g}(x,y) = g^{-1}(x,y)\Psi(x,y), \qquad (2.54)$$

with

$$g(x,y) = \tilde{g}(x,y)\cdot\left(J\tilde{g}(x,y)J^\dagger\right) = \begin{pmatrix} g_f(x,y)g_c(x,y) & 0 \\ 0 & g^{\rm c}_f(x,y)g^{\rm c}_c(x,y) \end{pmatrix}, \qquad (2.55)$$

the covariance of $\mathcal{D}\Psi(x,y)$,

$$\mathcal{D}\Psi^{\rm g}(x,y) = g^{-1}(x,y)\mathcal{D}\Psi(x,y), \qquad (2.56)$$

follows. In addition, since $\mathbf{d} + \tilde{A}^D(x,y) + J\tilde{A}^D(x,y)J^\dagger$ is Lorentz invariant, $\mathcal{D}\Psi(x,y)$ is transformed as a spinor just like $\psi(x,y)$ under a Lorentz transformation.

In order to obtain the Dirac Lagrangian for the fermion sector, the associated spinor one-form is introduced as the counterpart of (2.43):

$$\tilde{\Psi}(x,y) = \gamma_\mu\Psi(x,y){\rm d}x^\mu + {\rm i}\Psi(x,y)\chi. \qquad (2.57)$$

With the same inner products for the spinor one-forms as in [21]

$$\langle A(x,y){\rm d}x^\mu, B(x,y){\rm d}x^\nu\rangle = \bar{A}(x,y)B(x,y)g^{\mu\nu},$$
$$\langle A(x,y)\chi, B(x,y)\chi\rangle = -\bar{A}(x,y)B(x,y), \qquad (2.58)$$

and with vanishing other inner products, we obtain the Dirac Lagrangian

$$\mathcal{L}_D(x,y) = {\rm iTr}\langle\tilde{\Psi}(x,y), \mathcal{D}\Psi(x,y)\rangle$$
$$= {\rm Tr}\Big[{\rm i}\overline{\Psi}(x,y)\gamma^\mu(\partial_\mu + \tilde{A}^D_\mu(x,y) + J\tilde{A}^D_\mu(x,y)J^\dagger)$$
$$\times\Psi(x,y) - \overline{\Psi}(x,y)(\tilde{H}^D(x,y)$$
$$+J\tilde{H}^D(x,y)J^\dagger)\Psi(x,-y)\Big], \qquad (2.59)$$

where Tr is again the trace over internal symmetry matrices including the color and flavor symmetries and generation space. The total Dirac Lagrangian is obtained by summing (2.59) over y as follows:

$$\mathcal{L}_D(x) = \sum_{y=\pm}\mathcal{L}_D(x,y), \qquad (2.60)$$

which is apparently invariant under the Lorentz and gauge transformations.

With these preparations, we can apply the formulation proposed in this section to the reconstruction of the standard model.

3 Model construction

Based on the formulation proposed in the previous section, we now reconstruct the standard model including the YMH and fermion sectors. The reconstruction of the YMH sector is actually identical to that in [21] and for this reason we describe it only briefly.

3.1 The Yang–Mills–Higgs Lagrangian

The specifications of the flavor gauge fields $A_\mu(x,y)$, the color gauge field $G_\mu(x,y)$ and the Higgs boson field $\Phi(x,y)$ are presented in [21]. However, we here repeat those specifications, because they are crucially important in the reconstruction of the model based on our formulation in GDG.

We specify $A_\mu(x,y)$ in (2.17) by

$$A_\mu(x,+) = -\frac{\mathrm{i}}{2}\left\{\sum_{k=1}^{3}\sigma^k \otimes 1^4 A^k_{\mathrm{L}\mu} + aB_\mu\right\} \otimes 1^3, \quad (3.1)$$

$$A_\mu(x,-) = -\frac{\mathrm{i}}{2} bB_\mu \otimes 1^3, \quad (3.2)$$

where $A^k_{\mathrm{L}\mu}$ and B_μ are $SU(2)_\mathrm{L}$ and $U(1)$ gauge fields, respectively, and σ^k $(k = 1,2,3)$ are the Pauli matrices. 1^3 represents the unit matrix in the generation space and a and b are the $U(1)$ hypercharge matrices corresponding to $\psi(x,+)$ and $\psi(x,-)$ in (2.1), and expressed in 8×8 diagonal matrix form as

$$a = \mathrm{diag}\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},-1,\frac{1}{3},\frac{1}{3},\frac{1}{3},-1\right). \quad (3.3)$$

$$b = \mathrm{diag}\left(\frac{4}{3},\frac{4}{3},\frac{4}{3},0,-\frac{2}{3},-\frac{2}{3},-\frac{2}{3},-2\right). \quad (3.4)$$

$G_\mu(x,+) = G_\mu(x,-)$ in (2.18) is written by

$$G_\mu(x,\pm) = -\frac{\mathrm{i}}{2}\sum_{a=1}^{8}\sigma^0 \otimes \lambda'^a G^a_\mu \otimes 1^3, \quad (3.5)$$

where σ^0 is 2×2 unit matrix and λ'^a is 4×4 matrix obtained from the Gell-Mann matrix λ^a by adding a fourth line and column with all 0 entries:

$$\lambda'^a = \begin{pmatrix} & & & 0 \\ & \lambda^a & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.6)$$

This form is necessary to avoid interactions between leptons and color gauge fields. The Higgs field $\Phi(x,y)$ in (2.20) is represented in 24×24 matrix form by

$$\Phi(x,+) = \begin{pmatrix} \phi_0^* & \phi^+ \\ -\phi^- & \phi_0 \end{pmatrix} \otimes 1^4 \otimes 1^3,$$

$$\Phi(x,-) = \begin{pmatrix} \phi_0 & -\phi^+ \\ \phi^- & \phi_0^* \end{pmatrix} \otimes 1^4 \otimes 1^3. \quad (3.7)$$

Corresponding to (3.7), the symmetry breaking function $M(y)$ is given by

$$M(+) = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \otimes 1^4 \otimes 1^3, \quad M(-) = M(+)^\dagger. \quad (3.8)$$

With these specifications, the generalized field strength $\mathcal{F}(x,y)$ in (2.35) can be written explicitly and from these equations the YMH Lagrangian can be obtained, after rescaling the gauge and Higgs fields, as follows:

$$\mathcal{L}_{\mathrm{YMH}} = -\frac{1}{4}\sum_{k=1}^{3}\left(F^k_{\mu\nu}\right)^2 - \frac{1}{4}B^2_{\mu\nu} + |D_\mu h|^2$$
$$-\lambda(h^\dagger h - \mu^2)^2 - \frac{1}{4}\sum_{a=1}^{8} G^a_{\mu\nu}{}^\dagger G^{a\mu\nu}, \quad (3.9)$$

where the field strengths of flavor and color gauge fields are written

$$F^k_{\mu\nu} = \partial_\mu A^k_\nu - \partial_\nu A^k_\mu + g\epsilon^{klm}A^l_\mu A^m_\nu, \quad (3.10)$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad (3.11)$$

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu, + g_c f^{abc} G^{\mathrm{b}}_\mu G^{\mathrm{c}}_\nu, \quad (3.12)$$

and the covariant derivative of the Higgs boson is expressed as

$$D^\mu h = \left[\partial_\mu - \frac{\mathrm{i}}{2}\left(\sum_k \sigma^k g A^k_{\mathrm{L}\mu} + \sigma^0 g' B_\mu\right)\right] h, \quad (3.13)$$

with the following definition of the un-shifted Higgs boson h:

$$h = \begin{pmatrix} \phi^+ \\ \phi_0 + \mu \end{pmatrix}. \quad (3.14)$$

The Lagrangian in (3.9) is identical to the YMH Lagrangian of the standard model.

3.2 The fermion sector

Let us turn to the construction of the Dirac Lagrangian for the fermion sector. With the specifications of gauge and Higgs boson fields given by (3.1)–(3.8) and after the rescaling of the gauge and Higgs boson fields, we can obtain the explicit form of an covariant spinor one-form in (2.48). With the definition of the Dirac Lagrangian (2.60) and after some calculations regarding charge conjugation of the fermion fields, we find the Dirac Lagrangian for the standard model to be

$$\mathcal{L}_D = \sum_{y=\pm} \mathrm{i}\langle\tilde{\Psi}(x,y), \mathcal{D}\Psi(x,y)\rangle$$
$$= \mathrm{i}\overline{\psi}_\mathrm{L}(x)\gamma^\mu\left[1^8\partial_\mu - \frac{\mathrm{i}}{2}\left\{g\sum_{k=1}^{3}A^k_{\mathrm{L}\mu}\otimes 1^4 + g'\sigma^0 \otimes aB_\mu\right.\right.$$
$$\left.\left. + \; g_c\sum_{a=1}^{8}\sigma^0 \otimes \lambda'^a G_\mu\right\}\right] \otimes 1^3\psi_\mathrm{L}(x) + \mathrm{i}\overline{\psi}_\mathrm{R}(x)\gamma^\mu$$
$$\times\left[1^8\partial_\mu - \frac{\mathrm{i}}{2}\left\{g' bB_\mu + g_c\sum_{a=1}^{8}\sigma^0 \otimes \lambda'^a G_\mu\right\}\right]$$
$$\otimes 1^3\psi_\mathrm{R}(x) - \overline{\psi}_\mathrm{L}(x)h' \otimes 1^4 \otimes 1^3 g_\mathrm{Y}\psi_\mathrm{R}(x)$$
$$- \overline{\psi}_\mathrm{R}(x)g_\mathrm{Y}^\dagger h'^\dagger \otimes 1^4 \otimes 1^3\psi_\mathrm{R}(x), \quad (3.15)$$

where

$$\psi_\mathrm{L}(x) = \sqrt{2}\psi(x,+), \quad \psi_\mathrm{R}(x) = \sqrt{2}\psi(x,-), \quad (3.16)$$

and

$$h' = \begin{pmatrix} \phi_0^* + \mu & \phi^+ \\ -\phi^- & \phi_0 + \mu \end{pmatrix}. \quad (3.17)$$

The Yukawa coupling constant g_Y is explicitly given as

$$g_\mathrm{Y} = \mathrm{diag}(g^u, g^u, g^u, g^\nu, g^d, g^d, g^d, g^e), \quad (3.18)$$

where g^u, g^d, g^ν and g^e are complex Yukawa coupling constants written by 3×3 matrices in generation space. From the definitions of the fermion fields in (2.1), we easily find the Dirac Lagrangian in (3.15) to be the Lagrangian of the standard model.

4 Concluding remarks

Using the real structure operator J, we reformulated a previous formulation [21] in the GDG to incorporate the color gauge sector in the standard model in a simpler way. This was accomplished by replacing the fermion field $\psi(x, y)$ by $\Psi(x, y)$, which contains both the fermion and the antifermion field as in (2.4). Although the Lagrangians obtained in this way are identical to those obtained in our previous formulation of GDG, the expression of the generalized gauge field becomes very simple as in (2.9) and the basic formalism becomes very apparent. It would be very interesting to apply the method in this paper to the reconstruction of the left–right symmetric gauge model. Such a study will appear in a future paper.

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